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Nonlinear distortion of the Kelvin ship-wave pattern

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Linear wave theory is extensively used in research on the design of ship hull forms. Difficulty is being encountered, however, because of substantial differences between the calculated and measured phase geometry of the wave patterns generated. It seems likely that these differences may be at least partly due to nonlinear effects on phase velocity, and a nonlinear analysis of the Kelvin pattern has been undertaken as a basis for estimating the possible magnitude of such effects. It is noted that the Kelvin pattern due to a source in a finite tank contains a set of discrete free wave modes. An analysis of the nonlinear interactions in the general case of a steady multidirectional pattern of discrete cosine wave modes is undertaken, special attention being paid to the distortion of the phase anatomy, and the resulting theory is applied to the case of the Kelvin pattern in a tank. Sample computations using this analysis are discussed.

1. Introduction

The wave pattern generated by a travelling source as defined by Kelvin (1887) using linear theory has for a long time played a key role in ship-wave research. The waves generated by a ship can in fact be approximately represented as the sum of a series of elementary Kelvin source patterns and this principle is extensively used for studying the wave-making properties, especially for estimating the wave-making resistance of ships. Comparison of such theoretical estimates with experiment has shown that though there is generally fair qualitative agreement there is a consistent trend for the phases of the measured waves to lie forward of those predicted on the basis of linear Kelvin source arrays. These phase differences are causing difficulty in hull design investigations by the author using analysis of measured wave patterns behind ship models to deduce the locations of the principle sources of wave making.

In 1909 Hovgaard had already noted the tendency for bow wave crests to lie outside the theoretically predicted crest lines to an extent which varied with speed. Since that time, further evidence of this trend has been cited, for example, by Gadd (1969, 1971), Hogben (1957, 1971*a*), Inui (1963) and Newman (1970). Both Hovgaard and Gadd have suggested that this may be at least partly explained by outward displacement of the stream flow near the bow, following the shape of the waterlines, which is neglected in the linear theory; the latter (Gadd 1971) has in fact developed an approximate method of allowing for this effect. Support for this view may also be found in Shearer (1951) and Everest & Hogben (1970); Shearer showed that the phase discrepancy is negligible for fine formed mathematical models but Everest & Hogben, using variants of similar model form, showed that there is a discrepancy which increases as the bow waterline angle increases. This idea, however, does not satisfactorily account for the substantial speed dependence noted by Hovgaard (1909) and Hogben (1971*a*) and some further explanation is needed.

Gadd (1969) suggested that the effect may also be partly due to nonlinear influences on the wave propagation velocity and this idea was elaborated in the discussion by Lighthill, who proposed a simple empirical formula for assessing the magnitude of these influences. The author has examined this formula but has been unable to apply it because it is necessary to evaluate effective local wavenumbers, which he could not measure in practical experimental situations. He has also studied a number of other papers on nonlinear wave theory without finding any satisfactory basis for estimating the magnitude of these nonlinear phase shifts. He has therefore attempted to provide such a basis by examining second- and third-order interactions in the idealized case of the theoretical wave pattern due to a single Kelvin source in a tank of finite width. Such a single source can roughly represent a ship only at very high speeds but an approximate basis for estimating the phase shifts due to wave mode interactions has emerged which may reasonably be expected to indicate their order of magnitude at more realistic speeds also. The present paper according to its title is concerned with the investigation of the Kelvin pattern which is thought to be of some interest in its own right. Ship applications, though touched on, are not discussed in much detail.

The paper is in three main parts. The first is a discussion of the linear theory for waves due to a travelling source, special attention being paid to the phase anatomy both in open water and in a tank; in the latter case it is noted that a steady pattern of discrete free wave modes is formed downstream. The second considers how the free-surface condition for a steady multidirectional pattern of discrete wave modes may be satisfied to second order by an extension of Rayleigh's (1876) method for deriving the Stokes (1849) expansions for a single unidirectional wave train of permanent type. Third-order terms necessary for determining the second-order changes in wavenumber are included as in the Rayleigh analysis and are found to be rather numerous in the multidirectional case. The third part examines the application of this analysis to the case of a Kelvin wave source in a tank and discusses the bearing of the results on the practical problem of interpreting measured wave patterns as an aid to ship hull design.

2. Linearized theory for waves due to a travelling source

This section is concerned with linear analysis of waves due to a travelling source both in open water and in a tank, special attention being paid to the phase geometry. The theory for open water is already well documented but it will be briefly reviewed as an introduction to the physical concepts involved and a limiting case to be recovered for a tank of infinite width.

2.1. *Open water*

The name of Kelvin is coupled with the pattern due to a travelling source in open water because of the classic papers he wrote on the subject (Kelvin 1887, 1904). Since that time many other papers have been written on this topic including, notably, contributions by Havelock (1908, 1934) and Ursell (1960). Havelock (1934) has presented a simple physical interpretation according to which the pattern of free waves downstream is represented approximately as the sum of an infinite number of elementary sinusoidal wave trains propagating in all possible directions, each having an appropriate infinitesimal amplitude and a wavenumber satisfying the condition of steadiness with respect to the travelling source. On this basis he showed how the now familiar 'Kelvin patterns' of crest envelopes can be mapped by a simple graphical process.

A precise mathematical description of the complete pattern of surface elevation ζ_s, given, for example, by Lunde (1951), includes a so-called transient disturbance ζ_t in the neighbourhood of the source as well as the free wave contribution ζ_f extending over the whole downstream wave field and may be written for a source of strength m in deep water thus:

$$\zeta_s = \zeta_t + \zeta_f,$$

where
$$\zeta_t = \frac{4m}{\pi c}\int_0^{\frac{1}{2}\pi} \sec\theta\, d\theta \int_0^{\infty} \frac{e^{-\gamma z_m}(\sin\alpha x \cos\beta y)\,\gamma\, d\gamma}{\gamma - k\sec^2\theta}$$

(the infinite integral being interpreted as a principal value) and

$$\zeta_f = \frac{8km}{c}\int_0^{\frac{1}{2}\pi} e^{-\gamma z_m}\sec^3\theta\cos\alpha x\cos\beta y\, d\theta.$$

Here $k = g/c^2$, θ may be interpreted as the angle between the direction of elementary wave propagation and the direction of source motion, $\alpha = \gamma\cos\theta$, $\beta = \gamma\sin\theta$ and, in the expression for ζ_f, $\gamma = k\sec^2\theta$ and x and y are horizontal co-ordinates with origin above the source, which is travelling at depth z_m below the surface at speed c in the negative x direction. The double-integral expression for ζ_t can be evaluated numerically but care is necessary to exclude the neighbourhood of the poles at $\gamma = k\sec^2\theta$, which give rise to the free wave terms defined by the single-integral expression for ζ_t.

For the present investigation equations for mapping the crest envelopes will also be useful because they can effectively display the phase anatomy of the patterns. The term 'crest envelope' here denotes the construction described by Havelock (1934) in which the Kelvin pattern is generated by the crest lines of the component wave modes. An alternative picture of the pattern may be derived by tracing the lines of stationary phase as described, for example, by Lamb (1932). These show the effective phase due to the combination of the modes, and the resulting wave ridges are similar in shape to the crest envelopes but offset to the convex side of them. In particular, as was noted by Lamb, the stationary-phase argument leads to a phase difference of $\frac{1}{2}\pi$ between the transverse and diverging waves at the cusp, which does not occur in the case of the crest envelopes. In the present work crest envelopes are preferred because the

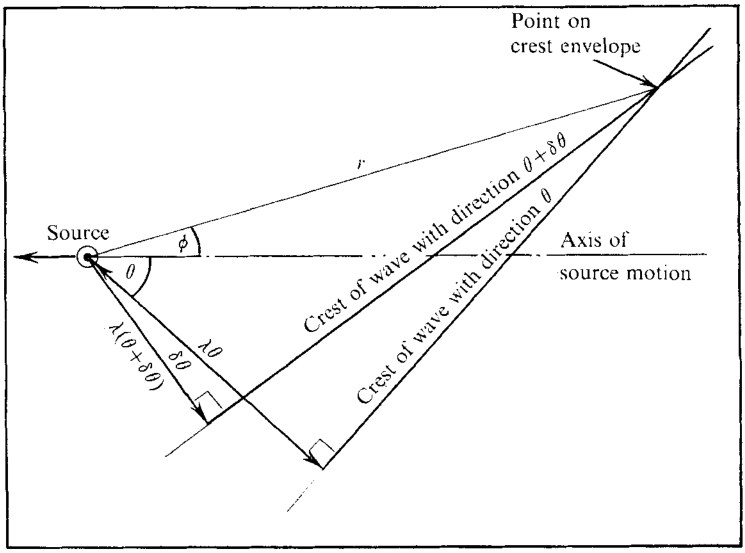

FIGURE 1. Diagram illustrating derivation of crest envelope.

author's practical concern is with the phases of the individual wave modes as determined by analysis of measured wave patterns and, in this context, stationary-phase lines are irrelevant. Another advantage is that with the aid of figure 1 exactly comparable results are easily derived for both the open-water and finite-tank-width cases whereas it is difficult to see how stationary-phase lines should be determined in the latter case.

The open-water crest-envelope formulae may be derived from the geometry of figure 1 as follows. The figure invokes Havelock's (1934) interpretation of the free wave pattern and shows the crest lines of two adjacent wave modes in the first cycle of the pattern whose intersection defines a point on the corresponding envelope. From inspection of this diagram it may be found that

$$r = \lambda_\theta \sec(\theta+\phi),$$

where

$$\lambda_\theta = 2\pi c^2 \cos^2\theta/g,$$

and hence

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{g}{2\pi c^2}\frac{d}{d\theta}\{\sec^2\theta\cos(\theta+\phi)\} = 0,$$

which leads to

$$\tan\phi = \tan\theta/(1+2\tan^2\theta).$$

It may be readily checked by writing $d(\tan\phi)/d\theta = 0$ that the maximum $\phi = \tan^{-1}(1/2\sqrt{2})$ occurs when $\theta = \tan^{-1}(1/\sqrt{2})$, thus confirming well-known properties of the Kelvin pattern, as described, for example, by Lamb (1932).

2.2. *Finite-width tank*

The waves due to a travelling source in a finite-width tank can be expressed in very similar terms to those for the open-water case except that the free wave patterns can contain only those discrete modes which satisfy the boundary conditions at the walls. The finite-channel situation has been investigated by

Srettensky (1936), Newman & Poole (1962) and Eggers (1962). Eggers has shown in various papers (Eggers 1962, 1963) on the analysis of waves generated by ship models that in a finite channel the wavenumbers γ_n of the admissible modes are defined by the positive real roots of

$$\gamma_n^2 = k\gamma_n \tanh \gamma_n h + (2\pi n/b)^2,$$

where $k = g/c^2$, h = depth of channel and b = breadth of channel.

Everest & Hogben (1969) have subsequently used the method of images to derive expressions defining the free wave pattern due to a source in a tank. These are closely analogous to the corresponding formulae for open water but since only the discrete modes prescribed by Eggers (1962) are admitted, sums replace integrals; thus, in deep water ($h \to \infty$)

$$\zeta_{tb} = \frac{4m}{\pi c} \sum_{n=-\infty}^{+\infty} \int_0^{\frac{1}{2}\pi} \sec\theta \, d\theta \int_0^\infty e^{-\gamma z_m} \frac{\sin \alpha x \cos\{\beta(y+nb)\} \gamma \, d\gamma}{\gamma - k\sec^2\theta}$$

(the infinite integral being interpreted as a principal value) and

$$\zeta_{fb} = \frac{8\pi m}{bc} \sum_{n=-\infty}^{\infty} \frac{e^{-\gamma_n z_m}}{2-\cos^2\theta_n} \cos\alpha_n x \cos\beta_n y, \tag{2.1}$$

where $\gamma_n = k\sec^2\theta$, $\alpha_n = \gamma_n \cos\theta_n$ and $\beta_n = 2\pi n/b = \gamma_n \sin\theta_n$, so that

$$\gamma_n^2 = \alpha_n^2 + \beta_n^2 = k\gamma_n + (2\pi n/b)^2.$$

Noting now that $2\pi n/b = k\sin\theta_n \sec^2\theta_n$ leads to $d\theta = 2\pi\cos^3\theta_n \, dn/bk(2-\cos^2\theta_n)$ it may be seen that the corresponding open-water result is recovered when $b \to \infty$.

The associated formulae for mapping the crest envelopes may be derived from the diagram, figure 1, used for the open-water case but noting again that only the admissible discrete wave modes as defined above are to be included. Thus in the figure it may be seen that, writing θ as θ_n and $\theta + \delta\theta$ as θ_{n+1},

$$r = \lambda_{\theta_n} \sec(\theta_n + \phi) = \lambda_{\theta_{n+1}} \sec(\theta_{n+1} + \phi), \tag{2.2}$$

where

$$\lambda_{\theta_n} = (2\pi c^2/g)\cos^2\theta_n,$$

so

$$\cos^2\theta_n \sec(\theta_n + \phi) = \cos^2\theta_{n+1} \sec(\theta_{n+1} + \phi),$$

which leads to

$$\tan\phi = \frac{\cos^2\theta_n \cos\theta_{n+1} - \cos^2\theta_{n+1}\cos\theta_n}{\cos^2\theta_n \sin\theta_{n+1} - \cos^2\theta_{n+1}\sin\theta_n}, \tag{2.3}$$

and an explicit equation for θ_n may be written as

$$\theta_n = \cos^{-1}\left(\frac{2}{1 + [1 + (16\pi^2/k^2b^2)\, n^2]^{\frac{1}{2}}}\right)^{\frac{1}{2}}. \tag{2.4}$$

Having seen that the linearized free wave pattern due to a source in a tank can be represented as the sum of a multidirectional series of discrete wave modes, we now examine the influence of second-order interactions between the modes of such a pattern.

3. Nonlinear interactions in steady multidirectional waves

For some years, oceanographers have become increasingly aware of the significant effect of nonlinear wave interactions on the process of wave generation by wind, and papers on this topic include important contributions by Longuet-Higgins (1962), Hasselmann (1961), Phillips (1960) and Barnett (1968). Oceanographic attention, however, has centred on third-order interactions which lead to effects which grow with time and are thus essentially unsteady phenomena. Some nonlinear analysis of the ship-wave problem has also been undertaken, for example, by Newman (1970), Howe (1967, 1968), Sisov (1961), Eggers (1966) and Gadd (1969). Newman (1970) examined the third-order solution for the Kelvin pattern and was led to the unexpected result that to this order there is some unsteadiness due to instability in the region of the diverging wave cusps. Howe examined nonlinear effects on the phase using the concept of slowly varying amplitude and direction but considered an artificial case involving only one wave train. The equations derived by Sisov (1961) define the nonlinear forces and moments but not the pattern. Eggers and Gadd allowed for second-order free-surface effects by introducing supplementary terms to neutralize surface pressure anomalies but phase distortions of the waves are not included in their analysis. In a discussion of Gadd's (1969) paper, Lighthill emphasized the importance of the phase shifts with which the present paper is particularly concerned and suggested an approximate interpretation for analysing them. However, as was mentioned in the introduction, the author was unable to apply this to actual experimental situations or to find any theory, other than that which is now proposed, which offers the possibility for a numerical evaluation of nonlinear phase distortion in a ship-like-wave pattern.

In this section nonlinear interactions in the general case of a multidirectional pattern of discrete cosine wave modes which is steady with respect to some travelling origin are considered. The analysis described extends Rayleigh's (1876) method for deriving the Stokes (1849) expansion for a simple unidirectional wave of permanent type to derive analogous results which may be applied to a steady multidirectional pattern.

The argument begins by defining a composite potential function in terms of the amplitudes and wavenumbers of a set of primary wave trains, all complying with the steadiness criterion, and a corresponding number of secondary trains due to the interactions. Then by satisfying the exact free-surface boundary conditions to second order the amplitudes of the secondary waves may be determined. Also, as in the Rayleigh analysis a second-order component of the wavenumber is determined by including certain third-order terms when applying the surface boundary conditions to the primary waves. This second-order contribution to the wavenumber has the effect of modifying the relation between the wavenumbers and directions of the primary waves corresponding to a given pattern speed. This is a crucial feature of the present analysis since it entirely determines the phase distortion of the Kelvin wave pattern due to the nonlinear interactions.

3.1. *Outline of the analysis*

The argument is modelled on Lamb's (1932) account of Rayleigh's (1876) analysis, but with some difference in notation. In this work the space co-ordinates are x longitudinally, y transversely and z vertically (positive downwards) with origin in the still water surface; θ denotes the angle between the direction of wave propagation and the x axis and c is the x component of phase velocity.

Consider the potential function

$$\phi = cx + \sum_{n=1}^{N} \phi_n + \sum_{r=1}^{N} \sum_{s=1}^{N} (\phi_{rs+} + \phi_{rs-}) \tag{3.1}$$

and the corresponding surface elevation

$$\zeta + \zeta_0 = \sum_{n=1}^{N} \zeta_n + \sum_{r=1}^{N} \sum_{s=1}^{N} (\zeta_{rs+} + \zeta_{rs-}), \tag{3.2}$$

where

$$\begin{aligned} \phi_n &= ca_n \cos\theta_n \exp[-(\alpha_n \sec\theta_n) z] \sin R_n, \\ \phi_{rs\pm} &= ca_{rs\pm} \cos\theta_{rs\pm} \exp\{-[(\alpha_r \pm \alpha_s) \sec\theta_{rs\pm}] z\} \sin(R_r \pm R_s), \\ \zeta_0 &= \text{mean level}, \quad \zeta_n = b_n \cos R_n, \\ \zeta_{rs\pm} &= b_{rs\pm} \cos(R_r \pm R_s), \quad R_n = \alpha_n (x + y \tan\theta_n). \end{aligned}$$

If ϕ_{rs+} and ϕ_{rs-} are to satisfy the Laplace continuity conditions, θ_{rs+} and θ_{rs-} must be given by

$$\theta_{rs\pm} = \tan^{-1}\left\{\frac{\alpha_r \tan\theta_r \pm \alpha_s \tan\theta_s}{\alpha_r \pm \alpha_s}\right\},$$

where a_n and b_n are primary wave amplitudes and $a_{rs\pm}$ and $b_{rs\pm}$ are secondary amplitudes to be determined.

In the analysis it is assumed that

$$\alpha_n a_n \quad \text{and} \quad \alpha_n b_n = O(\delta) \quad \text{as} \quad \delta \to 0$$

and that $\alpha_n a_{rs\pm}$ and $\alpha_n b_{rs\pm}$ are of $O(\delta^2)$. The free-surface pressure condition is

$$\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}_{z=\zeta} - 2g\zeta = 0 \tag{3.3}$$

and the kinematic surface condition is

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=\zeta} = \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}. \tag{3.4}$$

It may be shown (see Hogben 1971*b*, with some modification, noted below) that the pressure condition leads to

$$2a_n \alpha_n \cos\theta_n - 2gb_n/c^2 + G_n = 0, \tag{3.5}$$

where G_n denotes the third-order terms which arise when the expressions (3.1) and (3.2) for ϕ and ζ are substituted in the surface pressure equation,

$$2(\alpha_r \pm \alpha_s) a_{rs\pm} \cos\theta_{rs\pm} - 2gb_{rs\pm}/c^2 + \tfrac{1}{2} a_r a_s \alpha_r \alpha_s \{\cos(\theta_r - \theta_s) - 1\} - a_r a_s \alpha_r^2 = 0, \tag{3.6}$$

$$-2gb_{nn+} - a_n^2 \alpha_n^2 = 0. \tag{3.7}$$

The kinematic condition leads to

$$b_n - a_n = E_n \tag{3.8}$$

(where E_n denotes a set of third-order terms which arise when ϕ and ζ are substituted in the kinematic equation) and

$$(\alpha_r \pm \alpha_s)\, a_{rs\pm} - \tfrac{1}{2} a_r a_s \alpha_r^2 \sec\theta_r = (\alpha_r \pm \alpha_s)\, b_{rs\pm} \pm \tfrac{1}{2} a_r a_s \alpha_r \alpha_s \sec\theta_s \cos(\theta_r - \theta_s). \quad (3.9)$$

In the previous work (Hogben 1971*b*) the kinematic condition was neglected and it was assumed that $a_{rs\pm} = b_{rs\pm}$. As a result, though the potential function derived was in fact correct, the wave amplitudes were not. In addition only one of the numerous relevant third-order terms was included and the conclusions regarding phase distortion were thus incorrect.

The algebra involved in the derivation of the complete set of third-order terms denoted by G_n and E_n in (3.5) and (3.8) above is too lengthy to include in this paper but the principles of the analysis will be briefly indicated and the final results quoted. Details may be found in Hogben (1972).

In concept the method is still based on the argument used by Rayleigh for the single wave train but care has now been taken to ensure that all the additional third-order terms arising in the multidirectional case are included. Thus in defining ϕ and ζ it is noted that the primary terms denoted by the single summations in (3.1) and (3.2) should contain contributions from third-order terms of the type ϕ_{nss} and ζ_{nss} involving $\sin(R_n + R_s - R_s)$ and $\cos(R_n + R_s - R_s)$ and other relevant subscript pairings. It is now possible to compute both G_n and E_n by solving the equations (3.5) and (3.8) derived from the pressure and kinematic conditions taking account of all the third-order terms. These arise, for example, from expansion of the exponential factors and products of first- and second-order terms leading to the appropriate subscript pairing mentioned above. In this way the following results were obtained by Hogben (1972).

$$g/c^2 = \alpha_n \cos\theta_n (1 - H_n), \quad (3.10)$$

$$H_n = (k_n a_n)^2 + \sum_{\substack{s=1 \\ s \neq n}}^{N} h_{ns}, \quad (3.11)$$

where $k_n a_n = 2\pi a_n/\lambda_n$, λ_n is the wavelength of the nth primary wave mode and h_{ns} is a set of second-order terms which may be defined, using the abbreviations $a_\pm = (a_{ns\pm} + a_{sn\pm})$, $b_\pm = (b_{ns\pm} + b_{sn\pm})$ and $\alpha_\pm = \alpha_n \pm \alpha_s$, as follows:

$$\begin{aligned} 2a_n \alpha_n h_{ns} = {} & a_s \alpha_s^2 b_+ (\sec\theta_n + \sec\theta_s) + a_s \alpha_s^2 b_- (\sec\theta_n - \sec\theta_s) \\ & + a_s a_+ \alpha_+^2 (\sec\theta_n - \sec\theta_{ns+}) + a_s a_- \alpha_-^2 (\sec\theta_n - \sec\theta_{ns-}) \\ & + (a_s^2 \alpha_s a_n \alpha_n^2 \sec^2\theta_n + 2a_s^2 \alpha_s^2 a_n \alpha_n \sec\theta_s \sec\theta_n) \cos(\theta_n - \theta_s) \\ & - a_s \alpha_s b_+ \alpha_+ \sec\theta_{ns+} \cos(\theta_{ns+} - \theta_s) - a_s \alpha_s b_- \alpha_- \sec\theta_{ns-} \cos(\theta_{ns-} - \theta_s) \\ & - a_s \alpha_s a_+ \alpha_+ \sec\theta_n \{\cos(\theta_{ns+} - \theta_s) + 1\} \\ & - a_s \alpha_s a_- \alpha_- \sec\theta_n \{\cos(\theta_{ns-} - \theta_s) - 1\}. \end{aligned} \quad (3.12)$$

For completeness it may be of some interest to cite here also the expression for E_n derived by Hogben (1972), which may be written as

$$\frac{E_n}{a_n} = (k_n a_n)^2 \frac{c^2}{g} \left(\frac{7g}{8c^2} + \tfrac{1}{4} \alpha_n \cos\theta_n \right) + \sum_{\substack{s=1 \\ s \neq n}}^{N} e_{ns}, \quad (3.13)$$

where

$$2a_n\alpha_n e_{ns} = a_s\alpha_s^2\sec\theta_s(b_+-b_-)-a_sa_+\alpha_+^2\sec\theta_{ns+}-a_sa_-\alpha_-^2\sec\theta_{ns-}$$
$$-a_s\alpha_s b_+\alpha_+\sec\theta_{ns+}\cos(\theta_{ns+}-\theta_s)-a_s\alpha_s b_-\alpha_-\sec\theta_{ns-}\cos(\theta_{ns-}-\theta_s)$$
$$+\tfrac{1}{2}a_s^2a_n\alpha_n^3\sec^2\theta_n+a_s^2\alpha_s^2a_n\alpha_n\sec\theta_s\sec\theta_n\cos(\theta_n-\theta_s). \qquad (3.14)$$

In carrying out this analysis it was found that the quantities $a_{rs\pm}$ and $b_{rs\pm}$ are unchanged by the inclusion of third-order terms. From (3.6), (3.7) and (3.9) it may be found that

$$a_{rs\pm} = b_{rs\pm}+\frac{a_ra_s\alpha_r}{2(\alpha_r\pm\alpha_s)}\{\alpha_r\sec\theta_r+\alpha_s\sec\theta_s\cos(\theta_r-\theta_s)\} \quad \text{for} \quad r\neq s, \qquad (3.15)$$
$$a_{nn} = 0,$$

and, by using the abbreviation

$$U_{rs\pm} = g/c^2-(\alpha_r\pm\alpha_s)\cos\theta_{rs\pm},$$

that

$$b_{rs\pm} = \frac{a_ra_s\alpha_r}{2U_{rs\pm}}[\tfrac{1}{2}\alpha_s\{\cos(\theta_r-\theta_s)\mp 1\}+\cos\theta_{rs\pm}\{\alpha_r\sec\theta_r+\alpha_s\sec\theta_s\cos(\theta_r-\theta_s)\}-\alpha_r], \qquad (3.16)$$

$$b_{nn} = -\tfrac{1}{2}(c^2/g)(a_n\alpha_n)^2. \qquad (3.17)$$

By pairing the coefficients, writing $a'_{rs\pm} = (a_{rs\pm}\pm a_{sr\pm})$ and $b'_{rs\pm} = (b_{rs\pm}{}^{+}_{+}b_{sr\pm})$, it may be shown that the second-order solution, in which $H_n = 0$ so that

$$\sec\theta_n = c^2\alpha_n/g,$$

may be expressed in the alternative form

$$\phi = c\sum_{n=1}^{N} a_n\cos\theta_n\exp[-(\alpha_n\sec\theta_n)z]\sin R_n$$
$$+c\sum_{r=1}^{N}\sum_{s=1}^{r-1} a'_{rs+}\cos\theta_{rs+}\exp\{-[(\alpha_r+\alpha_s)\sec\theta_{rs+}]z\}\sin(R_r+R_s)$$
$$+c\sum_{r=1}^{N}\sum_{s=1}^{r-1} a'_{rs-}\cos\theta_{rs-}\exp\{-[(\alpha_r-\alpha_s)\sec\theta_{rs-}]z\}\sin(R_r-R_s), \qquad (3.18)$$

$$\zeta+\zeta_0 = \sum_{n=1}^{N} a_n\cos R_n+\sum_{r=1}^{N}\sum_{s=1}^{r}\{b'_{rs+}\cos(R_r+R_s)+b'_{rs-}\cos(R_r-R_s)\}, \qquad (3.19)$$

where
$$a'_{rs\pm} = \frac{a_ra_s\alpha_r\alpha_s}{2U_{rs\pm}}\{\cos(\theta_r-\theta_s)\mp 1\}, \qquad (3.20)$$

$$b_{rs\pm} = \frac{c^2}{g}[2(\alpha_r\pm\alpha_s)\cos\theta_{rs\pm}a'_{rs\pm}+\tfrac{1}{2}a_ra_s\alpha_r\alpha_s\{\cos(\theta_r-\theta_s)\mp 1\}-\tfrac{1}{4}a_ra_s(\alpha_r^2+\alpha_s^2)]$$
$$\text{for} \quad r\neq s, \qquad (3.21)$$

$$b'_{nn} = b_{nn} = -(c^2/2g)(a_n\alpha_n)^2. \qquad (3.22)$$

In the rudimentary case $N = 1$ it may be found from (3.10), (3.12) and (3.13) that this solution is in agreement with Rayleigh's results. In the case $N = 2$ a comparison may be made with the results of Longuet-Higgins (1962) for a single pair of wave trains. When the condition for steadiness is imposed on the formulae of Longuet-Higgins so that the notation corresponds as follows:

$$\sigma_1, \sigma_2 \equiv c\alpha_1, c\alpha_2, \quad |k_1 \pm k_2| \equiv \alpha_{12\pm}, \quad \theta = \theta_1 - \theta_2, \quad \psi_1, \psi_2 = R_1, R_2,$$

where Longuet-Higgins' symbols are on the left-hand side, it may be shown using (3.18)–(3.22) that the results are in agreement.

4. Application of nonlinear analysis to waves due to a travelling source

The analysis developed in the preceding section can in principle be applied to any steady multidirectional pattern of discrete cosine wave modes. In §2 it was seen that the waves due to a travelling source include a transient disturbance near the source and since this does not contain discrete modes it cannot be correctly taken into account in applying this second-order analysis. It was also noted that the free wave pattern for a source in open water is not strictly composed of discrete modes but can be approximated on this basis. For many practical purposes, in fact, the open-water case can be analysed as if it were the case of a tank of large width, provided that the correspondingly large number of modes does not overload the computer and that no intrusion is made into regions where reflexions from the imaginary walls are not negligible. The remainder of this section will therefore be concerned with waves due to a source travelling in a channel of finite width but infinite depth, and will not take account of interactions due to the transient disturbance near the source.

In §2, expressions were derived for the amplitudes and wavenumbers for all the modes in the linearized free wave pattern in a tank which become the primary waves in the second-order analysis. In theory the number of modes is infinite but in practice, provided that the speed is not too low or the tank width too large, the number need not be excessive. It must be appreciated however that when applying the interaction analysis the modes for $+\theta_n$ and $-\theta_n$ must be treated separately and not paired off as in the usual linear computations, since they interact with each other. This point may be more clearly understood by rewriting equation (2.1) for the free waves in a tank as

$$\zeta_{fb} = \frac{8\pi m}{bc} \sum_{n=-\infty}^{\infty} \frac{e^{-\gamma_n z_m}}{2-\cos^2\theta_n} \cos(\alpha_n x + \beta_n y)$$

and noting that each wave mode extends across the full width of the tank.

It is also important to notice that the formulae defining the admissible wavenumbers must be modified to allow for the second-order effects on the wavenumbers of the primary waves while still satisfying the boundary conditions at the walls. Strictly speaking the corresponding secondary waves should also be subject to these boundary conditions and this could be ensured by including relevant contributions from the primary waves of neighbouring image sources. In practice this would incur excessive computation but the difficulty may be avoided by keeping clear of regions affected by reflexion so that these conditions can be neglected. In this connexion it is important to appreciate that the discrete mode representation of the free waves is valid throughout the wave field, including regions upstream of the reflexion point. In most experimental situations the author has in fact found that there is a substantial working region containing discrete free waves in which both transient and reflexion effects may be neglected. Intuitively it must of course be expected that in this area there should be very little difference between the 'discrete' free waves and the corresponding 'continuous' open-water pattern.

On the basis of the foregoing assumptions application of the analysis described in §3 to the case of a source in a deep tank of finite width leads to the following expressions for the second-order free wave elevation ζ_{fb2}:

$$\zeta_{fb2} = \sum_{n=-N}^{+N} b_n \cos(\alpha_n x)\cos(\beta_n y)$$

$$+\sum_{\substack{r=-N \\ r\neq s}}^{+N}\sum_{s=-N}^{+N}\{b_{rs+}\cos(\alpha_+ x)\cos(\alpha_+ \tan\theta_{rs+} y)+b_{rs-}\cos(\alpha_- x)\cos(\alpha_- \tan\theta_{rs-} y)\}, \tag{4.1}$$

where $$b_n = \frac{8\pi m}{bc}\frac{e^{-\gamma_n z_m}}{2-\cos^2\theta_n},$$

$$\alpha_n = \gamma_n\cos\theta_n, \quad \beta_n = 2\pi n/b = \gamma_n \sin\theta_n$$

and $$\gamma_n = g\sec^2\theta_n/c^2(1-H_n),$$

so that $$\gamma_n^2 = \alpha_n^2+\beta_n^2 = \frac{g\gamma_n}{c^2(1-H_n)}+\left(\frac{2\pi n}{b}\right)^2. \tag{4.2}$$

$b_{rs\pm}$, $\alpha_\pm$, $\theta_{rs\pm}$ and H_n may be determined from the relevant formulae of §3, noting that $a_n = b_n - O(\delta^2)$.

Regarding the crest envelopes, equations (2.2) and (2.3) may be used but the values of θ_n defined by equation (2.4) must be modified to

$$\theta_n = \cos^{-1}\left(\frac{2}{1+[1+(16\pi^2/k^2b^2)(1-H_n)^2n^2]^{\frac{1}{2}}}\right)^{\frac{1}{2}}. \tag{4.3}$$

Because of the complicated dependence of H_n on θ_n defined by (3.11) and (3.12), equation (4.3) must be solved by iteration starting with the linear assumption $H_n = 0$.

A computer program has been written for calculating the free wave profiles to second order and the nonlinear distortion of the primary wave crest envelopes, and some sample results are shown in figures 2 and 3 and table 1. In all the cases shown, N has been taken as 6 so that the total number of primary wave modes corresponding to $-N \leqslant n \leqslant N$ is 13.

The concern of this paper is mainly with the nonlinear phase distortion but some brief comments may be made on the profiles in figure 2. These have been drawn for a source at depth $z_m = 0{\cdot}05b$ at a Froude number based on tank width $F_b = c/(gb)^{\frac{1}{2}} = 0{\cdot}5$. The second-order profiles are shown for a very high value of the source strength parameter $Q = 8\pi m/b^2c = 0{\cdot}08$ so that the influence of the nonlinearity can be easily seen. For actual ship wave patterns, in the author's experience, the equivalent Q value determined by comparing steepnesses of the component wave modes is generally less than about 0·01, as is explained below.

The curves display some slight bumpiness because of the limited number of modes but are otherwise physically reasonable. As expected, the nonlinear effect is greatest near the source and takes the form of a sharpening of the first crest and flattening of the first trough. It may be noted that a profile of ζ_t computed from linear theory is included in the figure for $Y = 0$ to show that the transient disturbance becomes negligible at a very short distance from the source. The corresponding profile for $Y = 0{\cdot}25$ was also computed but was found to be negligible.

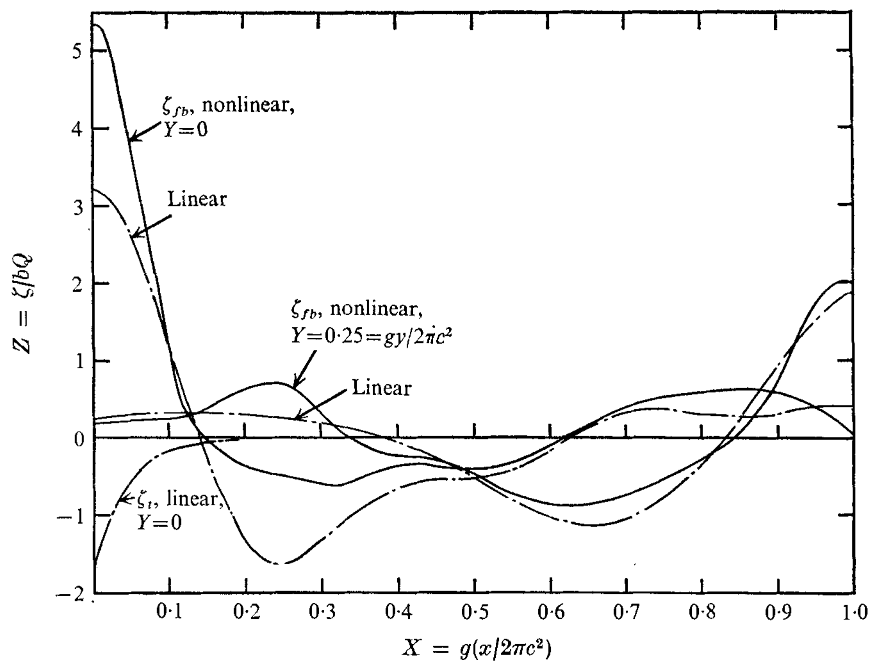

FIGURE 2. Sample wave profiles drawn for $F_b = 0{\cdot}5$ and $Z = 0{\cdot}05$ including second-order free waves for $Q = 0{\cdot}08$.

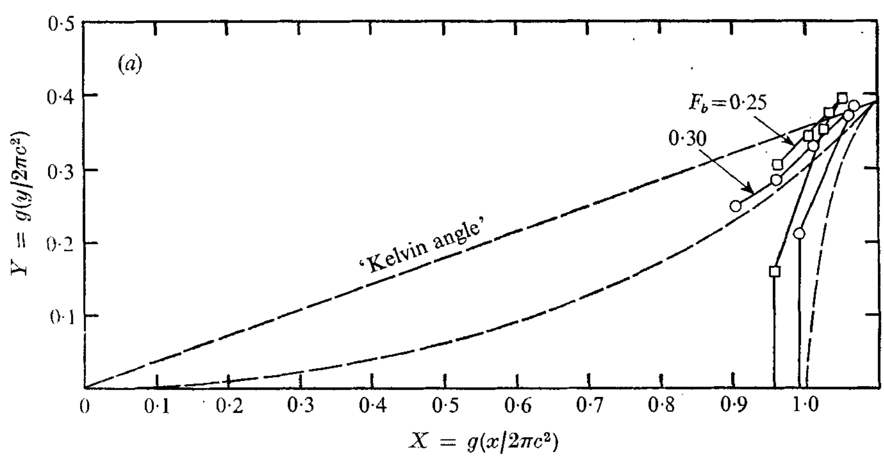

FIGURE 3 (a). For legend see facing page.

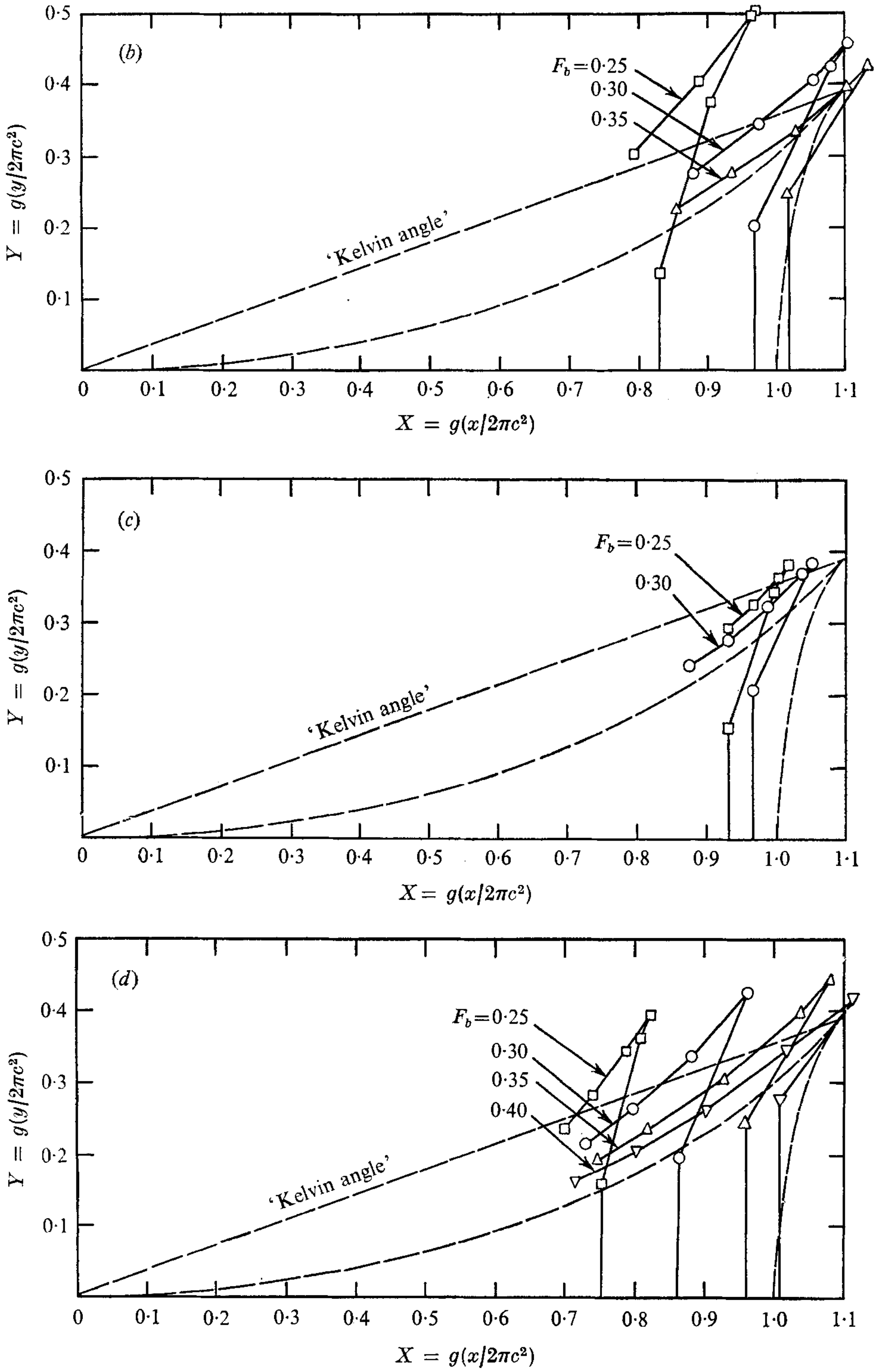

FIGURE 3. Nonlinear distortion of crest envelopes. (*a*) $Q = 0{\cdot}01$, $Z = 0{\cdot}025$. (*b*) $Q = 0{\cdot}02$, $Z = 0{\cdot}025$. (*c*) $Q = 0{\cdot}02$, $Z = 0{\cdot}05$. (*d*) $Q = 0{\cdot}04$, $Z = 0{\cdot}05$.

The nonlinear phase distortion of the primary waves is illustrated in figure 3 and table 1. A range of values of the parameters $Z = z_m/b$, F_b and Q is covered and in each case the figure shows the crest envelopes for the first cycle of the pattern. The points on the envelopes mark the points of intersection of the crests of adjacent modes. The dotted curves are the crest envelopes for the linear pattern in open water. Table 1 gives corresponding numerical details of the nonlinear distortion of the primary wavelengths and directions and includes a column showing values of $S_n = 2b_n/\lambda_n$, which define the steepnesses of the component modes.

The values of Z and F_b chosen effectively cover the normal working range of the author's experience. The corresponding values of Q were chosen to give an indication of the range over which significant distortions may be expected according to this theory. Approximate equivalence with actual ship-wave patterns may be established by comparing the values of S_n in table 1 with corresponding values derived from wave pattern measurements. The largest S_n which the author has so far recorded experimentally is 0·019 (taking modes for $+\theta_n$ and $-\theta_n$ separately as in the present paper) and this was in conditions approximating to $F_b = 0{\cdot}25$ and $Z = 0{\cdot}025$, for which results are shown for $Q = 0{\cdot}01$ in table 1. The maximum theoretical value of S_n in this case is 0·035 and it must thus be inferred that the equivalent Q for the experiment was much less than 0·01. Looking now at figure 3(*a*) it may be seen that the theoretical phase distortion for $Q = 0{\cdot}01$ is quite small, the maximum crest displacement being about 4% of the transverse wavelength. This is to be compared with phase shifts determined experimentally of the order of 10% of the transverse wavelength which may be found, for example, in Hogben (1971*a*). These results thus strongly suggest that the free wave interaction mechanism which has been described can only account for a relatively small part of the observed phase distortion.

It is of course possible to question the validity of the comparison. It could be, for example, that the experimentally determined values of S_n, based on far-field measurements, grossly underestimate the near-field wave steepness and the corresponding equivalent Q because of wave breaking. Another point which has a bearing on the validity of the comparison emerges from consideration of the speed dependence. The theoretical results in figure 3 show that the distortion diminishes with increases in speed and in table 1 it may be seen that this is to be expected because of the corresponding decrease of steepness due to the lengthening of the waves. The experimental results of Hogben (1971*a*), however, show increasing phase shift with increase of speed. The explanation for this probably lies in the influence of the interference between the bow and stern systems of an actual ship-wave pattern not occurring in the Kelvin pattern due to a single source. At the Froude numbers concerned, this interference was causing an increase in steepness and hence in the phase shift, owing to the increasing speed.

In spite of these reservations about the validity of the comparison it would still appear that the free wave interaction effects which have been investigated in this paper only account for a small part of the experimentally determined phase shifts. The remainder is probably mainly due to interaction between the free waves and the transient disturbance near the source which was omitted

			Q = 0·01, Z = 0·025			Q = 0·02, Z = 0·025			Q = 0·02, Z = 0·05			Q = 0·04, Z = 0·05		
F_b	n	θ_n (deg) Linear	θ_n (deg) Non-linear	S_n	H_n	θ_n (deg) Non-linear	S_n	H_n	θ_n (deg) Non-linear	S_n	H_n	θ_n (deg) Non-linear	S_n	H_n
0·25	0	0	0	0·0350	0·0413	0	0·0758	0·1709	0	0·04577	0·0677	0	0·09154	0·2490
	1	20·23	19·50	0·0340	0·0432	17·18	0·0732	0·1758	19·02	0·04165	0·0714	15·49	0·08330	0·2674
	2	33·29	32·25	0·0325	0·0496	28·74	0·0703	0·2041	31·62	0·03566	0·0793	26·23	0·07132	0·3009
	3	41·45	40·23	0·0317	0·0586	35·78	0·0684	0·2468	39·63	0·03035	0·0874	33·46	0·06070	0·3277
	4	46·98	45·67	0·0307	0·0676	40·57	0·0658	0·2833	45·12	0·02557	0·0940	38·80	0·05114	0·3430
	5	51·01	49·63	0·0296	0·0753	44·30	0·0625	0·3059	49·16	0·02125	0·0990	43·03	0·04250	0·3496
	6	54·10	52·69	0·0282	0·0814	47·46	0·0588	0·3160	52·29	0·01746	0·1027	46·48	0·03492	0·3505
0·30	0	0	0	0·0270	0·0094	0	0·0548	0·0301	0	0·04060	0·0339	0	0·08120	0·1373
	1	26·79	26·59	0·0262	0·0105	26·20	0·0531	0·0297	26·03	0·03674	0·0382	23·70	0·07348	0·1523
	2	40·64	40·29	0·0268	0·0174	39·49	0·0547	0·0556	39·65	0·03287	0·0484	36·32	0·06574	0·1934
	3	48·40	47·91	0·0276	0·0268	46·56	0·0565	0·0943	47·29	0·02918	0·0589	43·48	0·05836	0·2292
	4	53·43	52·82	0·0279	0·0359	51·08	0·0571	0·1286	52·26	0·02533	0·0672	48·42	0·05066	0·2494
	5	57·00	56·31	0·0276	0·0437	54·37	0·0564	0·1533	55·82	0·02151	0·0734	52·18	0·04302	0·2576
	6	59·71	58·97	0·0269	0·0499	56·99	0·0546	0·1681	58·54	0·01794	0·0780	55·17	0·03588	0·2581
0·35	0	0	0	0·0211	−0·0041	0	0·0417	−0·0218	0	0·03456	0·0116	0	0·06912	0·0416
	1	32·88	32·96	0·0213	−0·0039	33·41	0·0419	−0·0263	32·57	0·03191	0·0153	31·83	0·06382	0·0510
	2	46·60	46·55	0·0234	0·0028	46·62	0·0468	−0·0004	46·10	0·03035	0·0265	44·71	0·06070	0·0956
	3	53·77	53·58	0·0252	0·0117	53·16	0·0507	0·0361	53·13	0·02801	0·0376	51·29	0·05602	0·1357
	4	58·29	57·98	0·0262	0·0203	57·22	0·0528	0·0687	57·58	0·02491	0·0464	55·60	0·04982	0·1598
	5	61·46	61·08	0·0264	0·0276	60·12	0·0532	0·0931	60·72	0·02149	0·0530	58·81	0·04298	0·1717
	6	63·84	63·41	0·0260	0·0336	62·35	0·0524	0·1088	63·09	0·01812	0·0581	61·31	0·03624	0·1753
0·40	0	0	0	0·0169	−0·0097	0	0·0329	−0·0418	0	0·02910	−0·0007	0	0·05820	−0·0080
	1	38·28	38·48	0·0180	−0·0106	39·27	0·0352	−0·0502	38·24	0·02811	0·0020	38·34	0·05622	−0·0035
	2	51·42	51·49	0·0212	−0·0045	51·86	0·0421	−0·0255	51·20	0·02843	0·0131	50·67	0·05686	0·0417
	3	57·98	57·91	0·0236	0·0037	57·86	0·0473	0·0082	57·62	0·02710	0·0240	56·66	0·05420	0·0822
	4	62·05	61·88	0·0250	0·0116	61·53	0·0502	0·0386	61·61	0·02453	0·0327	60·50	0·04906	0·1081
	5	64·89	64·66	0·0255	0·0186	64·11	0·0513	0·0622	64·40	0·02143	0·0394	63·26	0·04286	0·1222
	6	67·00	66·73	0·0254	0·0242	66·08	0·0510	0·0782	66·50	0·01821	0·0447	65·41	0·03642	0·1288

TABLE 1. Nonlinear distortion of primary wavelengths and directions. $S_n = \gamma_n b_n/\pi = 2b_n/\lambda_n$, $H_n = -\delta\lambda_n/\lambda_n$.

from the theory because of the analytical difficulty of including it. In the actual ship model situation, the disturbance near the bow contains quite large lateral and forward components of velocity including those associated with the outward displacement of the stream flow studied by Gadd, as was mentioned in the introduction.

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REFERENCES

Barnett, T. P. 1968 *J. Geophys. Res.* **73**, 513.

Eggers, K. W. H. 1962 *Schiffstechnik*, **9**, 79.

Eggers, K. W. H. 1963 *Proc. International Seminar on Theoretical Wave Resistance.* Ann Arbor.

Eggers, K. W. H. 1966 *Proc. 6th ONR Symposium on Naval Hydrodynamics.* Washington.

Everest, J. T. & Hogben, N. 1969 *Trans. Roy. Instn. Nav. Archit.* **111**, 343.

Everest, J. T. & Hogben, N. 1970 *Trans. Roy. Instn. Nav. Archit.* **112**, 319.

Gadd, G. E. 1969 *Trans. Roy. Instn. Nav. Archit.* **111**, 487.

Gadd, G. E. 1971 *Nat. Phys. Lab. Ship Rep.* no. 156.

Hasselmann, K. 1961 *Proc. Conference on Ocean Wave Spectra.* Easton, Maryland.

Havelock, T. H. 1908 *Proc. Roy. Soc.* A**81**, 398.

Havelock, T. H. 1934 *Trans. Roy. Instn. Nav. Archit.* **76**, 430.

Hogben, N. 1957 *Trans. Roy. Instn. Nav. Archit.* **99**, 446.

Hogben, N. 1971*a* *Trans. Roy. Instn. Nav. Archit.* **113**, 345.

Hogben, N. 1971*b* *Nat. Phys. Lab. Ship Rep.* no. 159.

Hogben, N. 1972 *Nat. Phys. Lab. Ship Rep.* no. 167.

Hovgaard, G. W. 1909 *Trans. Roy. Instn. Nav. Archit.* **51**, 251.

Howe, M. S. 1967 *J. Fluid Mech.* **30**, 497.

Howe, M. S. 1968 *J. Fluid Mech.* **32**, 779.

Inui, T. 1963 *Trans. Soc. Naval Arch. Mar. Engrs*, **71**, 283.

Kelvin, Lord 1887 *Proc. Instn. Mech. Engrs*, **38**, 409.

Kelvin, Lord 1904 *Proc. Roy. Soc. Edin.* **25**, 311.

Lamb, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

Longuet-Higgins, M. S. 1962 *J. Fluid Mech.* **12**, 321.

Lunde, J. K. 1951 *Trans. Soc. Naval Arch. Mar. Engrs*, **59**, 25.

Newman, J. N. 1970 *Proc. 8th ONR Symposium on Naval Hydrodynamics.* Pasadena.

Newman, J. N. 1971 *J. Ship. Res.* **15**, 1.

Newman, J. N. & Poole, F. A. P. 1962 *Schiffstechnik*, **9**, 21.

Phillips, O. M. 1960 *J. Fluid Mech.* **9**, 193.

Rayleigh, Lord 1876 *Phil. Mag.* **5**, 257.

Shearer, J. R. 1951 *Trans. N.E. Coast Instn. Engrs Shipb.* **67**, 43.

Sisov, V. G. 1961 *Izv. Akad. Nauk SSSR Otd Tech. Mech. i Maschinostrojenie*, p. 75.

Srettensky, L. M. 1936 *Phil. Mag.* **22**, 1005.

Stokes, G. G. 1849 *Camb. Trans.* **8**, 441.

Ursell, F. 1960 *J. Fluid Mech.* **8**, 418.